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Bounding the size of certain rank 3 geometries with designs as rank 2 residues

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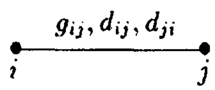
Abstract

We consider locally finite geometries of rank 3 belonging to certain diagrams where all strokes represent classes of non-trivial designs, at least one of them consisting of symmetric designs other than projective planes. The gonality diagram of such geometry is of spherical type, whereas the diameter diagram is of affine type. In cases like this, the criteria available from 24 cannot tell us anything about the finiteness or infiniteness of a geometry. In the main theorem of this paper we prove that, certain imply finiteness additional conditions on the parameters of the designs associated with the strokes of the diagram, a geometry is necessarily finite.

1. Introduction

We firstly recall some terminology and notation from [21]. As in [21], all geometries considered in this paper are residually connected and firm. Given a rank 2 geometry Δ over a pair of types $\{i,j\}$, we denote the gonality, the i-diameter and the j-diameter of Δ by g_{ij}, d_{ij} and d_{ji} respectively and we say that Δ is a (g_{ij},d_{ij},d_{ji})-gon. We represent the class of (g_{ij},d_{ij},d_{ji})-gons by the following diagram:



Note that a generalized m-gon is just a (m,m,m)-gon. In particular, a projective plane is a $(3,3,3)$-gon. A linear space other than a projective plane is a $(3,3,4)$-gon. A symmetric 2-(v,k,λ) design with $\lambda > 1$ is a $(2,3,3)$-gon.

We have $2 \leqslant g_{ij} \leqslant \min(d_{ij},d_{ji})$ and $|g_{ij} - g_{ji}| \leqslant 1$. Hence, $d_{ij} = d_{ji} = \infty$ if $g_{ij} = \infty$. If one of g_{ij} or g_{ji} is 2, then Δ is a generalized digon (that is, $g_{ij} = d_{ij} = d_{ji} = 2$).

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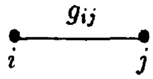
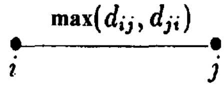
Let Γ be a geometry of rank $n \geqslant 3$ over a set of types I such that, for every choice of distinct types $i, j \in I$, all residues of Γ of type $\{i, j\}$ are (g_{ij}, d_{ij}, d_{ji})-gons for given g_{ij}, d_{ij}, d_{ji}. The *gonality diagram* $\mathscr{D}_{\text{gon}}$ and the *diameter diagram* $\mathscr{D}_{\text{diam}}$ for Γ are defined taking the following as ij-edges:



for $\mathcal{D}_{gon}$



for $\mathcal{D}_{diam}$

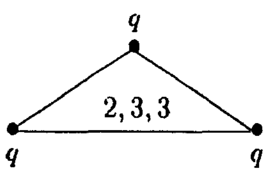
As both $\mathscr{D}_{\text{gon}}$ and $\mathscr{D}_{\text{diam}}$ look like Coxeter diagrams, we use for them the terminology and the notation normally used for Coxeter diagrams. If $\mathscr{D}_{\text{gon}} = A$ and $\mathscr{D}_{\text{diam}} = B$ for given Coxeter diagrams A and B, then we say that Γ is *placed between* A and B.

We say that Γ is *locally finite* if, for every type i, there is a positive integer s_i such that all flags of Γ of cotype i belong to at most $s_i + 1$ chambers. In particular, Γ is locally finite if it admits finite orders.

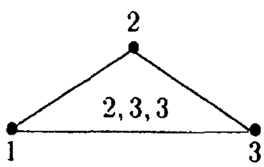
As a matter of fact, in many cases where upper bounds have been found for the size of a locally finite rank 3 geometry Γ with prescribed diagram, the geometry Γ is placed between a spherical and an affine diagram or between two spherical diagrams or it belongs to a spherical Coxeter diagram; see [3, Section 3], [20, 8, Theorem 2.11], [11, Theorem 37], [5, 16, Corollary 1], [11, Theorem 53], for instance.

On the other hand, there are geometries Γ of rank 3 placed between a spherical and an affine diagram that are locally finite but infinite (see [21]). Thus, we cannot claim that a locally finite geometry of rank 3 is finite whenever it is placed between a spherical and an affine diagram.

The above gave us a motivation to investigate rank 3 geometries placed between A_3 and $\tilde{A}_2$ in [21]. We considered the following diagram in [21], which we called $A_3 \mid \tilde{A}_2$:



where q is a finite order > 1. Thus, if we take the integers 1, 2, 3 as types as follows



then residues of elements of type 1 or 3 are projective planes of order q. The residue $\text{Res}(a)$ of an element a of type 2 has the following property: any two distinct elements of Res(a) of type i are always incident with some common element of Res(a) of type j ($\{i,j\} = \{1,3\}$), as $d_{13} = d_{31} = 3$, and there are distinct elements of Res(a) of type 1 both incident with two distinct elements of type 3, as $g_{13} = 2$. Note that every symmetric 2-($v, q+1, \lambda$) design with $\lambda > 1$ has these properties. However, there are many geometries that satisfy the above properties but are not symmetric designs.

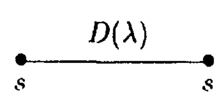
In [21] we have classified the flag-transitive geometries belonging to the above diagram and with the property that the residue Res(x) of an element x of type 1 or 3 is Desarguesian and that the stabilizer of x in the automorphism group of the geometry acts as a classical group in Res(x). We proved that there are only 4 simply connected examples with the above properties and that all of them are finite (and rather small, too). Their automorphism groups are $2^6.L_3(2)$, $U_3(3)$, $M_{22}.2$ and $U_4(3).2$ respectively. We have $q = 2$ in the first two examples and $q = 4$ in the last two. In each of them the residue of an element of type 2 is the 2-($q+2, q+1, q$) design of points and $q+1$-subsets of a $q+2$-set, with $\notin$ as incidence relation. The first example also admits an 8-fold quotient, where $2^3 : L_3(2)$ acts flag-transitively.

It is not clear what the result of [21] really means. Is some finiteness condition somehow implicit in the above diagram $A_3|\tilde{A}_2$? On the other hand, someone might object that the result of [21] only shows that the conditions assumed in [21] (in particular, the hypothesis that classical actions are induced on residues of elements of types 2 and 3) are almost impossible to satisfy and we are not allowed to hazard any conjecture on such a basis.

In this paper we consider some generalizations of $A_3|\tilde{A}_2$, but we always assume that residues of elements of type 2 are symmetric designs (as in the examples got in [21]). We obtain finite bounds for the size of a locally finite geometry belonging to one of those diagrams and satisfying certain additional hypotheses which either have nothing to do with groups or, if they ask something on automorphism groups, are not so heavy as in [21].

2. The diagram $D(\lambda, \mu, \nu; s)$

Henceforth, given positive integers λ and s, we represent the class of symmetric 2-($v, s+1, \lambda$)-designs as follows:

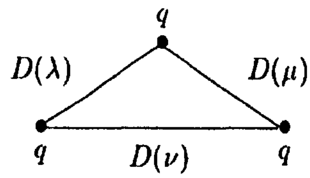


We recall that the number v of points (= number of blocks) of a symmetric 2-($v, s+1, \lambda$) design is $v = 1 + (s+1)s/\lambda$ (hence λ divides $(s+1)s$). Furthermore,

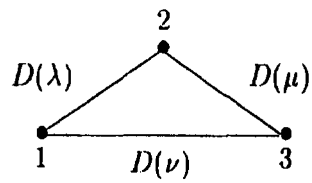
$\lambda < s+1$ (we do not include generalized digons among designs). The parameters s and λ will be called the *order* and the *multiplicity* of the design, respectively.

When $s = \lambda$ then the design is said to be *trivial*. It is easily seen that, for every positive integer s, there is just one trivial design of order s; its incidence graph is the complement of the $(s+2) \times 2$-grid graph.

By $D(\lambda, \mu, v; s)$ we mean the following diagram of rank 3:



We can always assume to have drawn the diagram is such a way that $\lambda \leqslant \mu \leqslant v$. We take 1, 2, 3 as types, labelling the nodes of the diagram as follows:



Note that $D(1,1,1;s)$ is just the affine diagram $\tilde{A}_2$ (with order s) and that, when $v > 1$, then $D(1,1,v;s)$ is placed between A_3 and $\tilde{A}_2$, as in [21] (actually, the examples for $A_3|\tilde{A}_2$ obtained in [21] belong to $D(1,1,s;s)$, with $s = 2$ and 4). When $1 < \mu$ ($\leqslant v$), then $D(1,\mu,v;s)$ is placed between the disconnected diagram $A_2 + A_1$

and $\tilde{A}_2$. When $1 < \lambda$ ($\leqslant \mu \leqslant v$), then $D(\lambda, v, \mu; s)$ is placed between the totally disconnected diagram of rank 3

and $\tilde{A}_2$. Note also that if $s = 1$, then $\lambda = \mu = v = 1$.

Let Γ be a geometry belonging to $D(\lambda, \mu, v; s)$. In order to extimate the size of Γ, we consider the following graph $\mathscr{G}$: the vertices of $\mathscr{G}$ are the elements of Γ of type 1, two such elements being adjacent in $\mathscr{G}$ precisely when they are incident with a common element of type 2. We call $\mathscr{G}$ the *(1,2)-graph* of Γ. The distance between two vertices a, b of $\mathscr{G}$ will be denoted by $d(a,b)$. We designate the diameter of $\mathscr{G}$ by d (note that we allow d to be ∞). We call d the *(1,2)-diameter* of Γ. It is easily seen that the diameter d_Γ of the chamber system of Γ satisfies the inequality $d_\Gamma \leqslant 4d + 3$ and we have $d_\Gamma = \infty$ if $d = \infty$. Therefore, a bound for the $(1,2)$-diameter d gives us a bound for d_Γ. We will prove the following:

Theorem 1. *If* $v > (s+1)/2$, *then* $d \leqslant 6$.

2.1. Proof of Theorem 1

Let a, b be vertices of the $(1,2)$-graph $\mathscr{G}$ of Γ at distance 2. For $i = 2,3$ we denote by I^i_{ab} the set of elements of Γ of type i incident with a and with some vertex of $\mathscr{G}$ adjacent to b.

Lemma 2. *Let a, b be vertices of $\mathscr{G}$ at distance 2. Then every element of I^i_{ab} is incident with at least v elements of I^j_{ab} for $\{i,j\} = \{2,3\}$.*

Proof. If $x \in I^2_{ab}$ and c is a vertex of $\mathscr{G}$ adjacent with b in $\mathscr{G}$ and incident with x in Γ, there are v elements of type 3 in the residue $\mathrm{Res}(x)$ of x incident with both c and a. This proves the statement of the lemma when $i = 2$ and $j = 3$.

Let $i = 3$ and $j = 2$. Given $u \in I^3_{ab}$, let c be a vertex of $\mathscr{G}$ adjacent with b in $\mathscr{G}$ and incident with u in Γ. Given an element x of type 2 incident with both b and c, there are v elements of type 3 in $\mathrm{Res}(x)$ incident with both b and c. Let v be one of them. Let C_{uv} be the set of the μ elements of type 2 in $\mathrm{Res}(c)$ that are incident with both u and v. We have two cases.

Case 1: At least one element $y \in C_{uv}$ is not incident with a.

In $\mathrm{Res}(y)$ there are v elements $c_1, c_2, \ldots, c_v$ of type 1 incident with both u and v. They are adjacent with b in $\mathscr{G}$ because they belong together with b to the design $\mathrm{Res}(v)$. For every $h = 1, 2, \ldots, v$ there are λ elements of type 2 in $\mathrm{Res}(u)$ incident with both c_h and a. If z is one of those elements of type 2, then $z \in I^2_{ab}$ and it is obtained from at most λ of the elements $c_1, c_2, \ldots, c_v$ because there are precisely λ elements of type 1 in $\mathrm{Res}(u)$ incident with both y and z. Therefore, there are at least v elements of I^2_{ab} in $\mathrm{Res}(u)$.

Case 2: All elements of C_{uv} are incident with a.

If $\lambda > 1$, then the same argument used in Case 1 works, but substituting λ with $\lambda - 1$. Let $\lambda = 1$. As each of the μ elements of C_{uv} is incident with a, we have $C_{uv} \subseteq I^2_{ab}$ and $\mu = 1$ (because $\lambda = 1$). Thus, if $v = 1$, then the statement is proved. Let $v > 1$. The element x of type 2 in $\mathrm{Res}(v)$ incident with both b and c (unique because $\lambda = 1$) is not incident with u because $C_{uv} \subseteq \mathrm{Res}(a)$ and $d(a,b) = 2$. Let w be an element of type 3 in $\mathrm{Res}(y)$ incident with both b and c and distinct from v (such an element exists because we have assumed $v > 1$). Let y and z be the elements of C_{uv} and C_{uw} respectively (these elements are uniquely determined because $\mu = 1$). We have $y \neq z$, otherwise y and x are distinct elements of type 2 incident with both v and w in $\mathrm{Res}(c)$, contrary to the fact that $\mu = 1$. As $C_{uv} \subseteq \mathrm{Res}(a)$ and $y \neq z$, in $\mathrm{Res}(u)$ we see that $z \notin \mathrm{Res}(a)$, because $\lambda = 1$. Therefore we can substitute C_{uv} with C_{uw}, thus going back to Case 1. $\square$

Lemma 3. *Let a, b be vertices of $\mathscr{G}$ at distance 2 and i = 2 or 3. Then* $1 + \nu(\nu - 1)/\mu \leqslant |I^i_{ab}|$.

Proof. Let $\{i,j\} = \{2,3\}$ and $x \in I^i_{ab}$. By Lemma 2 there are at least ν elements of I^j_{ab} incident with x and each of those elements is incident with at least $\nu - 1$ elements of I^i_{ab} other than x. Every element of $I^i_{ab} \cap \operatorname{Res}(a)$ that can be obtained in this way is obtained at most μ times. Hence $|I^i_{ab}| \geqslant 1 + \nu(\nu - 1)/\mu$. □

Lemma 4. *If $d \geqslant 7$, then there are vertices a, b, c of $\mathscr{G}$ with $d(a,c) = d(c,b) = 2$, $d(a,b) = 4$ and such that no element of I^2_{cb} is incident with any element of I^3_{ca}.*

Proof. Assume that for every choice of a, b, c as above some element of I^2_{cb} is incident with some element of I^3_{ca} and that nevertheless $d \geqslant 7$, if possible.

Let $a_0, a_1, \ldots, a_7$ be a path of $\mathscr{G}$ with $d(a_0, a_7) = 7$. By the previous assumptions, there is some $x \in I^2_{a_2,a_4}$ incident with some $u \in I^3_{a_2,a_0}$. We can assume to have chosen a_3 incident with x and a_1 incident with u. Similarly, there is $y \in I^2_{a_5,a_3}$ incident with some $v \in I^3_{a_5,a_7}$ and we can assume that $a_6 \in \operatorname{Res}(v)$. As $x \in I^2_{a_2,a'}$ for every element a' of type 1 incident with y and adjacent with a_3 in $\mathscr{G}$, we can always assume that a_4 is incident with y (if not, we can replace a_4 with an element a' as above). Let z be an element of type 2 incident with both a_3 and a_4. In $\operatorname{Res}(a_3)$ we can choose one element u' of type 3 incident with both x and z. Similarly, an element v' of type 3 can be chosen in $\operatorname{Res}(a_4)$ incident with both z and y. In $\operatorname{Res}(x)$, $\operatorname{Res}(z)$, $\operatorname{Res}(y)$ we can take elements a, b, c of type 1 incident with u and u', with u' and v' and with v' and v respectively. In $\operatorname{Res}(u)$, $\operatorname{Res}(u')$, $\operatorname{Res}(v')$ and $\operatorname{Res}(v)$ there are elements of type 2 incident with a_1 and a, with a and b, with b and c and with c and a_6 respectively. Therefore a_0, a_1, a, b, c, a_6, a_7 form a path of length 6 from a_0 to a_7, contrary to the assumption $d(a_0, a_7) = 7$. □

Lemma 5. *Let a, b, c be vertices of $\mathscr{G}$ with $d(a,c) = d(c,b) = 2$, $d(a,b) = 4$ and no element of I^2_{cb} incident with any element of I^3_{ca}. Then $\nu \leqslant (s+1)/2$.*

Proof. For every element x of $\operatorname{Res}(c)$ of type 2, let k^x_2 be the number of elements of I^3_{ca} incident with x and let $k_2 = \max(k^x_2 | x \in \operatorname{Res}(c), x \text{ of type } 2)$. Similarly, given an element $u \in \operatorname{Res}(c)$ of type 3, k^u_3 is the number of elements of I^2_{cb} incident with u and $k_3 = \max(k^u_3 | u \in \operatorname{Res}(c),\ u \text{ of type } 3)$. By Lemma 2 we have

(1) $\nu \leqslant k_2, k_3$

Let x be an element of $\operatorname{Res}(c)$ of type 2 such that $k^x_2 = k_2$. By our hypotheses on a, b, c we have $x \notin I^2_{cb}$. For every $y \in I^2_{cb}$, there are μ elements of type 3 in $\operatorname{Res}(c)$ incident with both x and y and each of them is obtained from at most k_3 elements $y \in I^2_{cb}$. Let U_x be the set of elements of type 3 in $\operatorname{Res}(c)$ incident with x and with some $y \in I^2_{cb}$.

By Lemma 3 we have $|U_x| \geqslant (v(v-1)+\mu)/k_3$. By the hypotheses we have made on a, b, c we have $U_x \cap I^3_{ca} = \emptyset$. Therefore:

(2) $k_2 + (v(v-1)+\mu)/k_3 \leqslant s+1$.

Similarly, starting from an element u of $\mathrm{Res}(c)$ of type 3 such that $k^u_3 = k_3$, we obtain

(3) $k_3 + (v(v-1)+\mu)/k_2 \leqslant s+1$.

By (2) and (3) we obtain

(4) $k_2 + \dfrac{(v(v-1)+\mu)k_2}{(s+1)k_2 - v(v-1) - \mu} \leqslant s+1.$

By (3) we get $k_2^2 - (s+1)k_2 + v(v-1) + \mu \leqslant 0$. Hence

(5) $2k_2 \leqslant s+1+\sqrt{(s+1)^2 - 4(v(v-1)+\mu)}$.

Using (5) and (1) it is not difficult to prove that $v \leqslant (s+1)/2$. □

End of the Proof of Theorem 1. Putting together Lemmas 4 and 5 we get Theorem 1.

2.2. More bounds

Theorem 6. *If* $v(v-1)+\mu > s(s+1-v)$, *then* $d \leqslant 4$.

Proof. Let $d(a,b) = 5$ for two vertices a, b of $\mathscr{G}$ and let c, d be adjacent vertices of $\mathscr{G}$ with $d(a,c) = d(d,b) = 2$. Given an element x of type 2 incident with both c and d, let U_c be the set of elements of type 3 incident with c, x and with some element of I^2_{ca} and let U_d be the set of elements of type 3 incident with d, x and with some element of I^2_{db}. As $d(a,b) = 5$, we have $U_c \cap U_d = \emptyset$. Indeed, let $u \in U_c \cap U_d$, if possible, and let $y \in I^2_{ca} \cap \mathrm{Res}(u)$ and $z \in I^2_{db} \cap \mathrm{Res}(u)$. There are elements of type 1 in $\mathrm{Res}(u)$ incident with both y and z, contrary to the assumption $d(a,b) = 5$.

Let us set $h = \min(|U_c|, |U_d|)$. As $U_c \cap U_d = \emptyset$ and since there are v elements of type 3 in $\mathrm{Res}(x)$ incident with both c and d, we have

(1) $h + v \leqslant s+1$.

On the other hand, $|I^2_{ca}| \geqslant 1 + v(v-1)/\mu$ by Lemma 3 and, for every element y of I^2_{ca}, there are μ elements of type 3 in $\mathrm{Res}(c)$ incident with both x and y. Each of them is obtained at most s times in this way. Therefore

(2) $h \geqslant (v(v-1)+\mu)/s$.

By (1) and (2) get $v(v-1)+\mu+sv \leqslant s(s+1)$. □

Theorem 7. *If* $2v(v-1)+\mu > s(s+1)$, *then* $d \leqslant 3$.

Proof. Let a, b, c be vertices of $\mathscr{G}$ with $d(a,b) = 4$ and $d(a,c) = d(b,c) = 2$. We have $I_{ca}^2 \cap I_{cb}^2 = \emptyset$ because $d(a,b) = 4$. By Lemma 3, we obtain $2v(v-1) + \mu \leqslant s(s+1)$. □

Theorem 8. *If* $v(v-1) + \mu v > s(s+1)$, *then* $d \leqslant 2$.

Proof. Let a, b, c be vertices of $\mathscr{G}$ with $d(a,b) = 3$, $d(a,c) = 2$ and c adjacent with b. Given an element x of type 2 incident with both c and b, there are v elements of type 3 in $\mathrm{Res}(x)$ incident with both c and b. As $d(a,b) = 3$, none of them is in I_{ca}^3. By Lemma 3 we obtain $v(v-1) + \mu v \leqslant s(s+1)$. □

Theorem 9. *If* $\mu(2v-1) > s(s+1)$, *then* $d = 1$.

Proof. Let a, b, c be vertices of $\mathscr{G}$ with $d(a,b) = 2$ and c adjacent with both a and b. Given elements x, y of type 2 incident with a and c and with c and b respectively, there are v elements of type 3 in $\mathrm{Res}(x)$ incident with both a and c and v elements of type 3 in $\mathrm{Res}(y)$ incident with both c and b. As $d(a,b) = 2$, the above shows that there are at least $2v$ elements of type 3 in $\mathrm{Res}(c)$. Therefore $2v \leqslant 1 + s(s+1)/\mu$. □

Corollary 10. *If* $v = \mu$ *and* $v(v+s) > s(s+1)$, *then* $d \leqslant 4$.

(trivial, by Theorem 6).

Corollary 11. *If* $v = \mu$ *and* $v(2v-1) > s(s+1)$, *then* $d = 1$.

(trivial, by Theorem 9).

Corollary 12. *If* $v = s = 2$, *then* $d \leqslant 4$.

(trivial, by Theorem 6).

Corollary 13. *If* $v = s > 2$, *then* $d \leqslant 3$.

(trivial, by Theorem 7).

Corollary 14. *If* $v = s$ *and* $\mu > 2$, *then* $d \leqslant 2$.

(trivial, by Theorem 8).

Corollary 15. *If* $v = s > 2$ *and* $\mu > (s+1)/2$, *then* $d = 1$.

(trivial, by Theorem 9).

3. Examples

3.1. Some sporadic examples

We start with a few remarks on the geometries of Theorem 1 of [21]. We have $\lambda = \mu = 1$ and $v = s$ ($= 2$ or 4) in those examples. Every geometry belonging to $D(1,1,s;s)$ is finite by Theorem 7 and Corollary 12. Thus, Lemma 11 of [21] (where it is proved that $v = s$) is the crucial step to prove the finiteness of an $A_3|\tilde{A}_2$ geometry satisfying the hypotheses of Theorem 1 of [21]. By that lemma and Lemma 9 of [21] (where it is proved that $s = 2$ or 4) we could already state that only a few and very small geometries could exist satisfying the hypotheses of Theorem 1 of [21]. Thus, it is not surprising at all that we could get all of them by coset enumeration.

Before to describe more examples for $D(\lambda,\mu,v;s)$ we need to state a bit of notation and a few definitions.

Let Γ be a flag-transitive geometry of rank 3 with types 1, 2 and 3 and let G be a flag-transitive group of type-preserving automorphisms of Γ. Given a chamber $C = (x_i)_{i=1}^3$ with x_i of type i, we denote by G_i the stabilizer of x_i in G, by G_{ij} the stabilizer $G_i \cap G_j$ in G of the flag $\{x_i, x_j\}$ ($1 \leqslant i < j \leqslant 3$) and by B the stabilizer of C. The triplet of subgroups $(G_i)_{i=1}^3$ satisfies the following conditions:

$$G_iG_j \cap G_iG_k = G_i(G_j \cap G_k) \quad \text{for } \{i,j,k\} = \{1,2,3\},$$
$$G_i = \langle G_{ij}, G_{ik}\rangle \quad \text{for } \{i,j,k\} = \{1,2,3\},$$
$$\langle G_i, G_j\rangle = G \quad \text{for } i,j = 1,2,3,\ i \neq j,$$
$$\bigcap_{g \in G} gBg^{-1} = 1.$$

A triplet $\mathcal{G} = (G_i)_{i=1}^3$ of subgroups of a group G satisfying the above conditions is said to be a (*geometric*) *parabolic system* of *rank* 3 in G. It is well known that, given a parabolic system $\mathcal{G} = (G_i)_{i=1}^3$ of rank 3 in a group G we can construct a geometry $\Gamma(\mathcal{G})$ admitting G as flag-transitive automorphism group, taking the right cosets of G_i as elements of type i of $\Gamma(\mathcal{G})$ ($i = 1,2,3$) and the relation having non-empty intersection as incidence relation see [13]. In particular, if G is a flag-transitive automorphism group of a given geometry Γ and the subgroups G_1, G_2, G_3 forming $\mathcal{G}$ are the stabilizers of the elements of a given chamber C of Γ, then $\Gamma(\mathcal{G})$ is a model of Γ.

We recall that, given a parabolic system $\mathcal{G} = (G_i)_{i=1}^3$ in a group G, the geometry $\Gamma(\mathcal{G})$ is simply connected if and only if the group G is the amalgamated product of the triplet $\mathcal{G}$, with amalgamation of the intersections $G_{ij} = G_i \cap G_j$ [18, pp. 234–236].

Let us now recall some definitions concerning designs. The *complement* of a non-trivial design Δ is the incidence structure with the same points and blocks as Δ but with $\notin$ as incidence relation instead of $\in$. If s and λ are the order and the multiplicity of Δ, then the complement of Δ is a non-trivial symmetric design of order $(s+1)(s-\lambda)/\lambda$

and multiplicity $\lambda + 1 + (s+1)(s-2\lambda)/\lambda$. A non-trivial symmetric design and its complement have the same automorphism groups with the same stabilizers of points and blocks, but different stabilizers of chambers (hence different parabolic systems).

A *biplane* of order s is a symmetric 2-$(v, s+1, 2)$ design ($v = s(s+1)/2$). Note that the trivial design of order 2 is the unique biplane of order 2 whereas all biplanes of order $s > 2$ are non-trivial (hence their complements can be considered). The complement of a biplane of order 3 is PG(2,2). Hence the complement of PG(2,2) is the unique biplane of order 3.

It is well known that $L_2(11)$ acts flag-transitively in a biplane of order 4, called the $L_2(11)$-*biplane*. Its complement is a symmetric 2-(11,6,3) design and $L_2(11)$ acts flag-transitively on it.

It is proved in [7] that there is only one flag-transitive biplane of order 5 with $\mathrm{PGL}_2(5)(=\mathrm{P\Gamma L}_2(4) = S_5)$ induced in a block and (dually) in the star of a point. The full automorphism group of this biplane is $2^4 : S_5$, but ASL(2,4) ($= 2^4 : A_5$) also acts flag-transitively in it. We call it the ASL(2,4)-*biplane*. We can construct a model for this biplane as follows. We take AG(2,4) as set of points. The blocks are the 16 hyperovals of PG(2,4) contained in AG(2,4) and defined in AG(2,4) by equations of the following form

$$(x+a)^2(y+b)^2 + (x+a)(y+b) + 1 = 0$$

with $a, b \in \mathrm{GF}(4)$. We can now describe some examples for $D(\lambda, \mu, \nu; s)$ other than those of [21].

(1) The group $U_3(3)$ admits a parabolic system (G_1, G_2, G_3) with

$$G_1 \cong G_2 \cong G_3 \cong L_3(2), \qquad G_{1,2} \cong G_{1,3} \cong G_{2,3} \cong S_4, \qquad B \cong S_3.$$

Let Γ_1 be the geometry defined by this parabolic system. The geometry Γ_1 belongs to $D(2,2,2;3)$ with all rank 2 residues isomorphic to the (unique) biplane of order 3 (namely, to the complement of PG(2,2)). We are not in the hypothesis of Theorem 1 (indeed we have $\nu = 2 = (s+1)/2$).

The geometry Γ_1 is the first member of a larger family Γ_1, Γ_2, Γ_3, Γ_4 with Γ_2, Γ_3, Γ_4 arising from J_2, $G_2(4)$ and 3.*Suz* respectively, of rank 4, 5 and 6 respectively. The residue of every element of Γ_i is isomorphic to Γ_{i-1} ($i = 2,3,4$).

(2) The group $U_3(5)$ admits a parabolic system (G_1, G_2, G_3) with

$$G_1 \cong G_2 \cong G_3 \cong A_7, \qquad G_{1,2} \cong G_{1,3} \cong G_{2,3} \cong L_3(2), \qquad B \cong \mathrm{Frob}(21).$$

The geometry defined by this parabolic system belongs to $D(4,4,4;7)$ with all rank 2 residues isomorphic to the complement of the design of points and planes of PG(3,2). As in the previous example, we are at the borders of Theorem 1: we have $\nu = 4 = (s+1)/2$.

(3) The Mathieu group M_{11} admits a parabolic system (G_1, G_2, G_3) with

$$G_1 \cong G_3 \cong L_2(11) \text{ and } G_2 \cong A_6, \qquad G_{1,2} \cong G_{1,3} \cong G_{2,3} \cong A_5, \qquad B \cong A_4.$$

Let Γ be the geometry defined by this parabolic system. The geometry Γ belongs to $D(2,2,4;4)$. Residues of elements of Γ of type 1 or 3 are isomorphic to the $L_2(11)$-biplane. The residues of the elements of type 2 are isomorphic to the trivial design of order 4. We now are in the hypotheses of Corollary 13. Hence the (1,2)-diameter of the universal cover of Γ is $\leqslant 3$. Actually, it can be checked using coset enumeration that M_{11} is the amalgamated product of the subgroups G_1, G_2, G_3 with amalgamation of the intersections $G_{i,j} = G_i \cap G_j$ $(1 \leqslant i < j \leqslant 3)$. Therefore Γ is simply connected.

(4) The Mathieu group M_{12} admits a parabolic system (G_1, G_2, G_3) with

$$G_1 \cong G_2 \cong G_3 \cong L_2(11), \qquad G_{1,2} \cong G_{1,3} \cong G_{2,3} \cong A_5, \qquad B \cong A_4.$$

The geometry defined by this parabolic system belongs to $D(2,2,2;4)$, with rank 2 residues isomorphic to the $L_2(11)$-biplane. The hypothesis of Theorem 1 does not hold in this case (we have $v = 2 < (s+1)/2 = 5/2$).

(5) We can find three copies G_1, G_2, G_3 of $2^4 : A_5$ inside an extension $2^{1+8} : 2^4 : A_5$ of $2^4 : A_5$ in such a way that any two of them intersect in a copy of A_5 and $G_1 \cap G_2 \cap G_3 = D_{10}$. The triplet (G_1, G_2, G_3) is a parabolic system in $G = 2^{1+8} : 2^4 : A_5$ and defines a geometry belonging to $D(2,2,2;5)$, with all rank 2 residues isomorphic to the ASL(2,4)-biplane. The hypothesis of Theorem 1 does not hold in this case (we have $v = 2 < (s+1)/2 = 3$).

(6) We can find a parabolic system (G_1, G_2, G_3) in $2^4 : A_7$ with

$$G_1 \cong G_3 \cong A_7 \quad \text{and} \quad G_2 \cong 2^3 : L_3(2) = AGL(3,2),$$
$$G_{1,2} \cong G_{2,3} \cong G_{3,1} \cong L_3(2), \qquad B \cong S_4.$$

Let Γ be the geometry defined by this parabolic system. The geometry Γ belongs to $D(3,3,6;6)$. Residues of elements of type 1 or 3 are isomorphic to the design of points and planes of PG(3,2). The residues of the elements of type 2 are isomorphic to the trivial design of order 6. We are in the hypothesis of Corollary 14. Therefore the universal cover of Γ has (1,2)-diameter $d \leqslant 2$. Actually, by coset enumeration it turns out that $2^4 : A_7$ is the amalgamated product of the subgroups G_1, G_2, G_3 with amalgamation of the intersections $G_{i,j} = G_i \cap G_j$ $(1 \leqslant i < j \leqslant 3)$. Therefore Γ is simply connected.

Problem. Find geometric constructions for the above examples (1)–(6) and for those of Theorem 1 of [21].

3.2. *Truncated D_4 buildings*

Let Δ be the D_4 building over GF(q). If we truncate the central node of the D_4 diagram, then we obtain a geometry Γ belonging to $D(\lambda,\lambda,\lambda;s)$ with $\lambda = q+1$ and $s = q(q+1)$. The rank 2 residues of Γ are isomorphic to the design of points and planes of PG($3,q$). The geometry Γ is simply connected by Theorem 1 of [19] and because buildings are 2-simply connected. Furthermore, Γ is finite, with $(q^3+1)(q^2+1)(q+1)$

elements of each type and the (i,j)-diameter of Γ is 2 (with i, j any two types; see Section 2). However, the hypothesis of Theorem 1 does not hold in Γ.

We can do the same as above starting from a Coxeter complex of type D_4, obtaining by truncation a simply connected geometry belonging to $D(2,2,2;2)$. Rank 2 residues are now isomorphic to the trivial design of order 2 and we are in the hypotheses of Corollary 12. The (i,j)-diameter is 2, less than the upper bound stated in Corollary 12.

The previous construction can easily be generalized to produce geometries of arbitrary rank n with all rank 2 residues isomorphic to the design of points and planes of $\mathrm{PG}(3,q)$ or to the trivial design of order 2. For instance, starting from the affine diagram $\tilde{D}_4$ and truncating the central node of the diagram we obtain examples of rank 4.

3.3. The geometry $\Delta(s,3)$

Given integers $s \geqslant 2$ and $n \geqslant 2$, the complement of the $(s+n) \times n$-grid graph is a geometry of rank n where all residues of rank 2 are isomorphic with the trivial design of order s. We denote this geometry by $\Delta(s,n)$. Note that all residues of $\Delta(s,n)$ of rank m $(2 \leqslant m < n)$ are isomorphic to $\Delta(s,m)$. All truncations of $\Delta(s,n)$ of rank m $(2 \leqslant m \leqslant n)$ are isomorphic to $\Delta(s+n-m,m)$.

The geometry $\Delta(s,3)$ belongs to $D(s,s,s;s)$. The (i,j)-diameter of $\Delta(s,3)$ is 1 (with i, j any two types). If $s > 2$, then $\Delta(s,3)$ satisfies the hypothesis of Corollary 15. When $s = 2$ then we are in the hypothesis of Corollary 12.

Proposition 16. *Let $s > 2$. Then $\Delta(s,3)$ is the unique geometry belonging to the diagram $D(s,s,s;s)$.*

The restriction $s > 2$ is essential in this proposition. Indeed the truncation of the D_4 Coxeter complex (Section 3.2) belongs to $D(2,2,2;2)$ but it is not isomorphic to $\Delta(s,2)$.

We need to state a bit of notation and a few lemmas before to prove the above proposition.

Henceforth Γ is a geometry belonging to $D(s,s,s;s)$ with $s \geqslant 3$. By Corollary 15, any two elements of type i are incident with some common element of type j, for every choice of distinct types $i,j = 1,2,3$. For $\{i,j,k\} = \{1,2,3\}$ and distinct elements a, b of type i, let w be an element of type k incident with both a and b. In $\mathrm{Res}(w)$ we find s elements of type j incident with both a and b. Therefore, any two distinct elements of type i are incident with at least s common elements of type j.

Given incident elements a, u of type 1 and 3 respectively, we designate by $p_a(u)$ the unique element of type 2 in $\mathrm{Res}(a)$ that is not incident with u. Similarly, given incident elements a, x of type 1 and 2, we denote by $q_a(x)$ the unique element of type 3 in $\mathrm{Res}(a)$ that is not incident with x. Trivially, we have $q_a(p_a(u)) = u$ and

$p_a(q_a(x)) = x$ for every a of type 1 and every choice of u and x in $\mathrm{Res}(a)$ of types 3 and 2 respectively.

Lemma 17. *Let a, b be elements of type 1 and let u, v be distinct elements of type 3 incident with a and b, respectively. Then $p_a(u) \neq p_b(v)$.*

Proof. The statement is obvious when $a = b$. Let $a \neq b$ and let $p_a(u) = p_b(v)$, if possible. Let us denote $p_a(u)(= p_b(v))$ by x and let y be an element of type 2 incident with both a and b and different from x (such an element exists, as there are at least s elements of type 2 incident with both a and b). As $y \neq x = p_a(u) = p_b(v)$, y is incident with both u and v.

There are at least s elements of type 3 incident with both x and y in $\mathrm{Res}(a)$ and at least s elements of type 3 incident with both x and y in $\mathrm{Res}(b)$. Since none of u and v is incident with x and y is incident with u, v and with precisely s other elements of type 3, there are just s elements of type 3 incident with both x and y and all of them are incident with both a and b. In $\mathrm{Res}(y)$ we now see that v is not incident with a and u is not incident with b. Let now w be any of the elements of type 3 incident with both x and y (hence with both a and b, by the above). As $s \geqslant 3$, there is an element c of type 1 in $\mathrm{Res}(w)$ incident with both x and y and other than a and b. In $\mathrm{Res}(y)$ we see that c is incident with both u and v. Thus u and v are distinct elements of $\mathrm{Res}(c)$ of type 3 not incident with b, contrary to the fact that $\mathrm{Res}(c)$ is a trivial design. □

Lemma 18. *Given an element u of type 3 and distinct elements a, b of type 1 in $\mathrm{Res}(u)$, we have $p_a(u) = p_b(u)$.*

Proof. Let us set $y = p_a(u)$ and $z = p_b(u)$ and let $y \neq z$ if possible. Then $y \notin \mathrm{Res}(b)$ and $z \notin \mathrm{Res}(a)$. Let x be an element of $\mathrm{Res}(u)$ of type 2 incident with both a and b. There are s elements of type 3 in $\mathrm{Res}(a)$ incident with both x and y and s elements of type 3 in $\mathrm{Res}(b)$ incident with both x and z and u is not one of them. As $2s+1 > s+2$, there is at least one element v of type 3 incident with all of a, b, x, y, z. In $\mathrm{Res}(v)$ there are $s-1$ elements of type 1 incident with all of x, y, z. Let c be one of those elements. We have $c \neq a, b$ because $a \notin \mathrm{Res}(z)$ and $b \notin \mathrm{Res}(y)$. As $y = p_c(q_c(y))$, we have $u = q_c(y)$ by Lemma 17. That is, u is incident with c and we have $y = p_c(u)$. Similarly, u is incident with c and we have $z = p_c(u)$. Hence $y = z$, contrary to the hypothesis that $y \neq z$. □

By Lemma 18 we can define a function p from the set of elements of type 3 to the set of elements of type 2 by the clause $p(u) = p_a(u)$, with a any element of type 1 incident with u.

Lemma 19. *The function p is bijective.*

Proof. The function p is injective by Lemma 17. It is also surjective, because $x = p(q_a(x))$ for every element x of type 2, with a any element of type 1 incident with x. □

Lemma 20. *Given an element x of type 2, the function p maps the set of elements of* $\mathrm{Res}(x)$ *of type* 3 *onto the set of elements of Γ of type* 2 *other than x.*

Proof. For every element y of type 2 other than x, we have $y = p(q_a(x))$, with a any element of type 1 incident with both x and y. The lemma is now quite evident. □

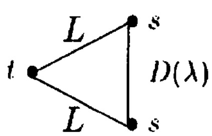
End of the Proof of Proposition 16. By Lemmas 19 and 20, the geometry Γ has $s+3$ elements of each type. It is now straightforward to prove that $\Gamma \cong \Delta(s,3)$.

Problem. Only a finite number of examples exist for $D(2,2,2;2)$, by Corollary 12. We know two of them, namely $\Delta(2,3)$ and the truncated Coxeter complex of type D_4. Are there any more examples? Note that $\Delta(2,3)$ has 5 elements of each type, hence it is not a quotient of the truncated Coxeter complex of type D_4, which has 8 elements of each type (and it is simply connected, by Theorem 1 of [19]). Is $\Delta(2,3)$ simply connected?

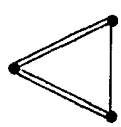
Remark. The diagram $D(1,1,1;1)$ is just the thin case of $\tilde{A}_2$. The Coxeter complex of this type is the unique simply connected geometry belonging to $D(1,1,1;1)$. It admits infinitely many quotients.

4. The diagram LD $(s; s, t)$

By $\mathrm{LD}(\lambda; s, t)$ we mean the following diagram of rank 3:



where t, s are finite orders with $1 \leqslant t \leqslant s$, λ is a positive integer $\leqslant s$, L denotes the class of linear spaces (here, 2-$(st+t+1, t+1, 1)$ designs) and $D(\lambda)$ has the meaning stated in Section 2. The gonality diagram associated to the above diagram is A_3 or $\tilde{A}_2$ according to whether $\lambda > 1$ or $\lambda = 1$. When $s = t$ then the diameter diagram is $\tilde{A}_2$ and we have a special case of $A_3|\tilde{A}_2$. When $s > t$ then the diameter diagram is as follows (not even of affine type):



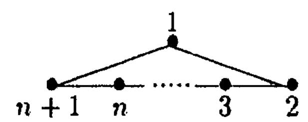
We are mainly interested in the special case LD($s;s,t$) of LD($\lambda;s,t$), where the vertical edge of the diagram represents the trivial design of order s. Note that LD($s;s,s$) is the special case $D(1,1,s;s)$ of $A_3|\tilde{A}_2$, which includes the geometries of Theorem 1 of [21]. Note also that LD($1;s,s$) $= D(1,1,1;s) = \tilde{A}_2$.

4.1. Examples

Before to examine LD($s;s,t$) we give some examples for LD($\lambda;s,t$). In many of them we have $\lambda = s$. However, we also have examples with $\lambda < s$.

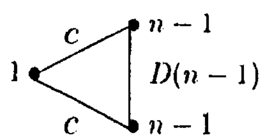
(1) As we have remarked above, the geometries of Theorem 1 of [21] belong to LD($s;s,s$).

(2) *Truncated buildings of type* $\tilde{A}_n$. Let Δ be a building of type $\tilde{A}_n$, with residues of elements isomorphic to PG(n,q) ($n \geqslant 3$):



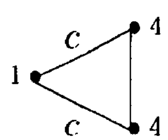
Here $1,2,\ldots,n,n+1$ are the types. Let Γ be the truncation of Δ obtained by dropping all elements of type $3,4,\ldots,n$. The geometry Γ belongs to LD($\lambda;s,t$) with $\lambda = (q^{n-1}-1)/(q-1)$, $s = q\lambda$ and $t = q$. Note that Γ is infinite and that for each quotient of Δ we get a quotient of Γ. Furthermore, Γ is simply connected, by Theorem 1 of [19].

The above can be repeated with a Coxeter complex of type $\tilde{A}_n$, obtaining a simply connected geometry for LD($n-1;n-1,1$):



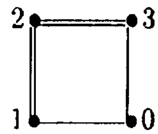
The symbol c denotes the class of circular spaces. The vertical edge of the diagram here represents the trivial design of order $n-1$.

(3) *Two geometries for* $U_4(3)$ *and* He. Two flag-transitive geometries are mentioned in [2, Examples 51 and 53] belonging to $LD(1;4;1)$:



They admit $U_4(3)$ and He respectively as flag-transitive automorphism groups. The gonality diagram is $\tilde{A}_2$. Hence the universal covers of these geometries are infinite, by [22].

(4) *An exceptional geometry for* McL. The McLaughlin group acts flag-transitively on a finite geometry Δ_1 belonging to the following diagram [23]



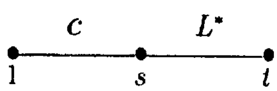
where 0, 1, 2, 3 are types. The residues of the elements of Δ_1 of types 1 and 3 are isomorphic to the flat C_3 geometry for the alternating group A_7 and Δ_1 has order 2 at each type. The geometry Δ_1 is 2-simply connected [23], even if it belongs to a non-spherical Coxeter diagram and is finite.

If we truncate the elements of type 0, then we obtain a geometry Γ_1 belonging to LD(3;6,2), with types 1, 2, 3 (inherited from Δ_1). The residues of the elements of Γ_1 of type 2 are isomorphic to the design of points and planes of PG(3,2), whereas residues of elements of type 1 or 3 are isomorphic to the linear space of points and lines of PG(3,2). As Δ_1 is 2-simply connected, Γ_1 is simply connected, by Theorem 1 of [19]. Trivially, McL acts flag-transitively in Γ_1, too.

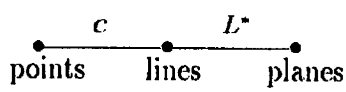
Another flag-transitive 2-simply connected geometry Δ_2 with the same diagram as Δ_1 has been constructed by Li [15]. That geometry is infinite and its residues of type C_3 are isomorphic to the flat C_3 geometry for A_7, as in Δ_1. We can play with Δ_2 the same truncation game as with Δ_1, thus obtaining a flag-transitive simply-connected geometry Γ_2 with the same diagram LD(3;6,2) as Γ_1. However, Γ_2 is infinite, as Δ_2 is infinite. This makes it clear that the diagram LD(3;6,2) does not involve any finiteness information.

4.2. *Unfolding* $c.L^*$ *and folding* LD(s; s, t)

Let Δ be a geometry belonging to the following diagram, which we call $c.L^*$:



where 1, s, t are finite orders and L^* designates the class of dual linear spaces. Let us call the elements of Δ points, lines and planes, according to their types:



It is easily seen that the following properties are equivalent in Δ:

(**LL**) the point-line system of Δ is a partial plane;
(**LL***) the plane-line system of Δ is a partial plane.

Let **LL** or **LL*** hold in Δ. Then we can form a new geometry Γ over the set of types $\{1,2,3\}$ as follows. The planes of Δ are the elements of Γ of type 1. The pairs (a,i) with a a point of Δ and $i = 2$ or 3 are the elements of Γ of type i and we state that two pairs $(a,2)$ and $(a,3)$ are incident in Γ if and only if $a \neq b$ and there is a line of Δ (unique by **LL**) incident with both a and b. A plane u of Δ and a pair (a,i) are declared to be incident in Γ precisely when the point a is incident with u in Δ. It is not difficult to check that Γ belongs to $\mathrm{LD}(s;s,t)$. We call Γ the *unfolding* of Δ and we denote it by $\mathrm{Unf}(\Delta)$.

It is clear that the point-plane system of Δ is isomorphic with the system of elements of $\mathrm{Unf}(\Delta)$ of type 2 and 1 (or 3 and 1). Furthermore, let δ be the function fixing every plane of Δ and mapping every pair (a,i) onto the pair (a,j) ($\{i,j\} = \{2,3\}$ and a a point of Δ). The function δ is an involutory automorphism of $\mathrm{Unf}(\Delta)$ permuting the types 2 and 3. Furthermore, x and $\delta(x)$ are never incident in $\mathrm{Unf}(\Delta)$, for every element x of $\mathrm{Unf}(\Delta)$ of type 2 or 3.

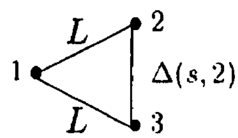
Many geometries exist belonging to the diagram $c.L^*$ and satisfying **LL** (equivalently, **LL***); some of them are mentioned in [2]; more examples are described in [10]. We only give a few examples here.

The affine geometry $\mathrm{AG}(3,2)$ belongs to $c.L^*$ with $s = t = 2$. Trivially, **LL** holds in it. Its unfolding is the 8-fold quotient of the geometry for $2^6 : L_3(2)$ of Theorem 1 of [21].

The Steiner system $S(22,6,3)$ for the Mathieu group M_{22} belongs to $c.L^*$ with $s = t = 4$ and satisfies **LL**. Its unfolding is the geometry for M_{22} considered in Theorem 1 of [21].

A *semibiplane* [12] is just the point-plane system of a geometry belonging to $c.L^*$ with $t = 1$ and satisfying **LL** (note that every biplane is a semibiplane). Hence every semibiplane (in particular, every biplane) can be unfolded and it gives us a geometry belonging to $\mathrm{LD}(s;s,1)$, where $s+2$ is the size of its planes (blocks).

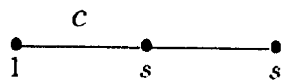
Conversely, let Γ be a geometry belonging to LD(s; s, t), with types 1–3 as follows:



and let Γ admit an involutory automorphism δ fixing all elements of type 1, permuting the types 2 and 3 and such that $\delta(x)$ and x are never incident, for every element of Γ of type 2 or 3. Then we can construct a rank 3 geometry Δ, as follows. The elements of Γ of type 1 and 2 are respectively the planes and the points of Δ. The lines of Δ are the flags of Γ of type $\{2,3\}$. A plane and a point or a line of Δ are incident in Δ if they are incident as elements or flags of Γ. A point a and a line $\{b,c\}$ of Δ, with b of type 2 and c of type 3, are incident in Δ if either $a = b$ or $\delta(a) = c$. It is easy to check that Δ belongs to $c.L^*$ with orders s, t at the second and third node of the diagram and that **LL** holds in Δ. Hence Δ can be unfolded; it is quite evident that $\mathrm{Unf}(\Delta) = \Gamma$. We call Δ the *folding* of Γ and we denote it by $\mathrm{Fl}(\Gamma)$. Thus, a geometry Γ belonging to $\mathrm{LD}(s;s,t)$ *can be folded* if and only if it admits an involutory automorphism δ with the above properties.

Trivially, $\mathrm{Unf}(\Delta)$ can be folded for every $c.L^*$ geometry Δ satisfying **LL** and we have $\mathrm{Fl}(\mathrm{Unf}(\Delta)) = \Delta$.

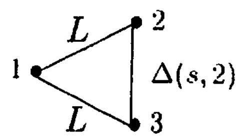
Needless to say, not every $\mathrm{LD}(s;s,t)$ geometry can be folded. For instance, if Γ belongs to $\mathrm{LD}(s;s,s)$ and it can be folded, then $\mathrm{Fl}(\Gamma)$ belongs to the following diagram



There are just three geometries belonging to this diagram [9], namely the tetrahedron, $\mathrm{AG}(3,2)$ and $S(22,6,3)$ (with pairs of points as lines). Therefore, the only geometries belonging to $\mathrm{LD}(s;s,s)$ that can be folded are $\Delta(1,3)$ (which is the unfolding of the tetrahedron), the 8-fold quotient of the geometry for $2^6 : L_3(2)$ of [21] and the geometry for M_{22} of [21].

4.3. *Bounding the size of certain* $\mathrm{LD}(s;s,t)$ *geometries*

In this section Γ is a geometry belonging to $\mathrm{LD}(s;s,t)$. We take 1, 2, 3 as types, as follows:



We denote by $\mathrm{Tr}_i(\Gamma)$ the rank 2 truncation of Γ obtained by dropping the elements of type i (with $i = 2$ or 3).

As residues of elements of Γ of type 1 are now trivial designs, for every element a of type 1 we can define a function p_a mapping every element x of $\mathrm{Res}(a)$ of type i onto the unique element of $\mathrm{Res}(a)$ of type j not incident with x, for $\{i,j\} = \{2,3\}$. Given an element x of type $i = 2$ or 3 and distinct elements a, b of $\mathrm{Res}(x)$ of type 1, let $q_x(a,b)$ be the (unique) element of type 3 (respectively, 2) in $\mathrm{Res}(x)$ incident with both a and b. Trivially, $q_x(a,b) \neq p_a(x), p_b(x)$. Therefore, if $p_a(x) = p_b(x)$, then $\mathrm{Tr}_i(\Gamma)$ is not a partial plane. Conversely, let a, b be distinct elements of type 1 and let x, y be distinct elements of type i incident with both a and b. Trivially, $q_x(a,b)$ is not incident with y. Therefore $q_x(a,b) = p_a(y) = p_b(y) \neq q_y(a,b)$. Similarly, $q_y(a,b) = p_a(x) = p_b(y)$. We have thus proved the following:

Lemma 21. *Given any two distinct elements a, b of type* 1, *there are at most two elements of type i (= 2 or 3) incident with both of them; if x, y are two distinct elements of type i incident with both a and b, then there are just two elements of type j (= 3 or 2 respectively) incident with both a and b, we have* $p_a(x) = p_b(x) = q_y(a,b)$

and $p_a(y) = p_b(y) = q_x(a,b)$ and these are the two elements of type j incident with both a and b.

Corollary 22. *The rank 2 geometry* $\mathrm{Tr}_2(\Gamma)$ *is a partial plane if and only if* $Tr_3(\Gamma)$ *is a partial plane.*

(trivial, by Lemma 21).

Lemma 23. *Let Γ be flag-transitive. Then Γ can be folded if and only if* $\mathrm{Tr}_i(\Gamma)$ *is not a partial plane ($i = 2$ or 3).*

Proof. If Γ can be folded, then $\mathrm{Tr}_i(\Gamma)$ is not a partial plane, as it is isomorphic with the plane-point system of $\mathrm{Fl}(\Gamma)$, which is not a partial plane.

Conversely, let $\mathrm{Tr}_i(\Gamma)$ be not a partial plane. Assume that $i = 2$. Given an element x of type 2, we define an equivalence relation $\equiv_x$ on the set of elements of $\mathrm{Res}(x)$ of type 1 stating that $a \equiv_x b$ iff $p_a(x) = p_b(x)$. Let G be a flag-transitive automorphism group of Γ. The stabilizer G_x of x in G preserves $\equiv_x$. On the other hand, it acts flag-transitively on the linear space $\mathrm{Res}(x)$. Since the flag-transitivity in linear spaces implies the primitivity on the set of points [5](2.3.7), the relation $\equiv_x$ is either the identity relation or the trivial relation. As G is flag-transitive, either $\equiv_x$ is the identity for all x of type 2 or it is trivial for all x of type 2. In the first case $\mathrm{Tr}_2(\Gamma)$ is a partial plane by Lemma 21, contrary to our hypotheses. Therefore $\equiv_x$ is always trivial. By Corollary 22, we can permute the types 2 and 3, obtaining the same conclusion as above. Therefore, for every element x of type $i = 2$ or 3 there is just one element $\delta(x)$ of type $j = 3$ or 2 respectively, such that x and $\delta(x)$ are not incident but they are incident with the same elements of type 1. It is now clear that Γ can be folded. □

We designate by d_i the diameter of the collinearity graph of $\mathrm{Tr}_i(\Gamma)$ ($i = 2$ or 3), with the elements of Γ of type 1 taken as points. It is clear that Γ is finite if and only if any of d_2 and d_3 is finite.

Theorem 24. *Let Γ be flag-transitive. Then either* $\mathrm{Tr}_2(\Gamma)$ *and* $\mathrm{Tr}_3(\Gamma)$ *are partial planes or $d_2, d_3 \leqslant 2 + (s-t)/2$.*

Proof. Let $\mathrm{Tr}_i(\Gamma)$ be not a partial plane, $i = 2,3$. Then Γ can be folded by Lemma 23. The geometry $\mathrm{Tr}_i(\Gamma)$ is isomorphic to the plane-point system of $\mathrm{Fl}(\Gamma)$, which has diameter $\leqslant 2 + (s-t)/2$ (see [4]). □

Flag-transitive infinite geometries belonging to $\mathrm{LD}(s;s,t)$ exist. Some examples of this kind with $t = 1$ have been described in Section 4.1(2). On the other hand there are flag-transitive $\mathrm{LD}(s;s,t)$ geometries Γ that are finite and where nevertheless $\mathrm{Tr}_i(\Gamma)$ is a partial plane. For instance, the geometry for M_{22} is the only one of the four geometries of Theorem 1 of [21] that can be folded. Therefore $\mathrm{Tr}_i(\Gamma)$ is a partial plane if Γ is any of the three remaining geometries mentioned there, by Lemma 23. Nevertheless, those geometries are finite (and even very small).

4.4. *The diagram* LD$(s;s,s)$

By Theorem 1, for every integer $s \geqslant 2$ there are only finitely many (possibly none) geometries belonging to LD$(s;s,s)$ $(= D(1,1,s;s))$. Actually, we know only five simply connected geometries belonging to LD$(s;s,s)$ with $s > 1$: the four geometries mentioned in Theorem 1 of [21] for $2^6 : L_3(2)$, $U_3(3)$, M_{22}, $U_4(3)$ and another one, discovered by Pasechnik [17] using coset enumeration, defined by the following parabolic system in $3 \times L_3(2)$:

$$G_2 \cong G_3 \cong \mathrm{Frob}(21), \qquad G_1 \cong A_4, \qquad G_{1,2} \cong G_{1,3} \cong G_{2,3} \cong 3, \qquad B = 1,$$

(the types are as in Section 4.3). Generators $s_i \in G_{jk}$ $(\{i,j,k\} = \{1,2,3\})$ can be chosen in such a way to obtain the following presentation for $3 \times L_3(2)$:

$$\begin{aligned}
&s_1^3 = s_2^3 = s_3^3 = 1, \qquad s_2s_1^{-1}s_2^{-1} = s_1^{-1}s_2s_1^{-1}, \qquad s_3s_1^{-1}s_3^{-1} = s_1^{-1}s_3s_1^{-1},\\
&(s_1s_2^{-1})^7 = (s_1s_3^{-1})^7 = 1, \qquad (s_2s_3)^2 = (s_3s_2)^2 = 1,\\
&s_2s_3^{-1}s_2 = s_3s_2^{-1}s_3 = s_2^{-1}s_3s_2^{-1}.
\end{aligned}$$

Note that the geometry for $2^6 : L_3(2)$ also admits $2^6 : \mathrm{Frob}(21)$ as flag-transitive automorphism group with a parabolic system apparently similar to the above. Starting from that parabolic system we obtain the following presentation for $2^6 : \mathrm{Frob}(21)$:

$$\begin{aligned}
&s_1^3 = s_2^3 = s_3^3 = 1, \qquad s_2s_1^{-1}s_2^{-1} = s_1^{-1}s_2s_1^{-1}, \qquad s_3s_1^{-1}s_3^{-1} = s_1^{-1}s_3s_1^{-1},\\
&(s_1s_2^{-1})^7 = (s_1s_3^{-1})^7 = 1, \qquad (s_2s_3^{-1})^2 = (s_3^{-1}s_2)^2 = 1,\\
&s_2s_3s_2 = s_3s_2s_3 = s_2^{-1}s_3^{-1}s_2^{-1}.
\end{aligned}$$

Theorem 25. *Let Γ be a simply connected flag-transitive geometry belonging to* LD$(s;s,s)(= \mathrm{D}(1,1,s;s))$ *with $s > 1$. Then Γ is one of the five examples mentioned above.*

Proof. Let G be a flag-transitive automorphism group of Γ. Given the nodes of LD$(s;s,s)$ types as in Section 4.3, residues of elements of types 2 and 3 are projective planes of order s, whereas the residues of the elements of type 1 are isomorphic to the trivial design of order s. By [14] we have one of the following:

(i) the residues of the elements of Γ of types 2 and 3 are classical projective planes of order s and the action induced by G in them contains $L_3(s)$;

(ii) for every element x of types 2 or 3, the stabilizer of x in G acts in Res(x) as a Frobenius group of order $(s+1)(s^2+s+1)$; the number s is even, we have $s+1 \equiv 0 \pmod 3$ and s^2+s+1 is prime.

Case (i) has been examined in [21] and it gives us the four simply connected geometries of Theorem 1 of [21]. Thus, we assume that (ii) occurs.

Chosen a chamber $C = (x_i)_{i=1}^3$ with x_i of type i, let $(G_i)_{i=1}^3$ be the parabolic system defined by C in G and let $B = \bigcap_{i=1}^3 G_i$ be the stabilizer of C in G. We have $B = 1$ [21, 3.1]. Therefore G_i acts faithfully in $\mathrm{Res}(x_i)$. As $\mathrm{Res}(x_1) = \Delta(s,2)$, the flag-transitivity of G_1 in $\mathrm{Res}(x_1)$ is equivalent to the 2-transitivity of G_1 on the $s+2$ elements of $\mathrm{Res}(x_1)$ of type i, for $i = 2$ or 3. As $B = 1$, the group G_1 is sharply 2-transitive on those elements. Therefore, $s+2$ is a prime power. Since s is even, we have $s+2 = 2^n$ for some positive integer n. On the other hand, $s+1 \equiv 0 \pmod 3$. Therefore n is even, $n = 2m$ say, and $s \equiv 2 \pmod 4$. On the other hand, if $m \geqslant 2$, then $s/2 \equiv 3 \pmod 3$. Hence there is some prime p dividing $s/2$ with an odd exponent and such that $p \equiv 3 \pmod 4$, contrary to the well known Bruck–Ryser condition on order of finite projective planes. Therefore $m = 1$. Hence $s = 2$ and we have $G_2 \cong G_3 \cong \mathrm{Frob}(21)$, $G_1 \cong A_4$, $G_{ij} \cong 3$ for $1 \leqslant i < j \leqslant 3$ (and $B = 1$). It is straightforward to check that there are just two possible ways two amalgamate two copies G_2 and G_3 of Frob(21) with a copy G_1 of A_4 in such a way that the intersections $G_i \cap G_j$ $(1 \leqslant i < j \leqslant 3)$ are pairwise distinct and all isomorphic to the ciclic group of order 3. These two ways are described by the two sets of relations given before. Using coset enumeration it is checked that those two sets lead to grous of order $2^3.3^2.7$ and $2^6.3.7$ respectively. The first one can be identified as $3 \times L_3(2)$. The second one is the subgroup $2^6 : \mathrm{Frob}(21)$ of $2^6 : L_3(2)$, flag-transitive on the LD(2; 2, 2) geometry of $2^6 : L_3(2)$. $\square$

5. Concluding remarks

Given a diagram $\mathscr{D}$ of rank 3, we say that $\mathscr{D}$ is of *finite type* if there is a positive integer d such that, for every geometry Γ belonging to $\mathscr{D}$, the chamber system of Γ has diameter $\leqslant d$. On the other side, if the chamber system of every simply connected geometry belonging to $\mathscr{D}$ has diameter ∞, then we say that $\mathscr{D}$ is of *infinite type*. In an intermediate situation, every locally finite geometry belonging to $\mathscr{D}$ is finite but $\mathscr{D}$ is not of finite type. In this case we say that $\mathscr{D}$ is of *nearly finite type*.

For instance, if $\mathscr{D}_{\mathrm{gon}}$ is non-spherical, then $\mathscr{D}$ is of infinite type, except possibly for a few degenerate cases ([22]; see also [19]). On the other hand, if $\mathscr{D}_{\mathrm{diam}}$ is spherical, then $\mathscr{D}$ is of finite type [24]. Thus, one might conjecture that, apart from some exceptions over which we hopefully will be able to get control, the diagrams of nearly finite type are essentially those placed between a spherical and an affine diagram. However, we cannot honestly claim that the results obtained in this paper and the examples we have described give a strong support to the above conjecture; some counterexamples described in [21] also warn us that things are perhaps not so easy as that conjecture pretends.

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